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# The Dissipative Linear Boltzmann Equation for Hard Spheres

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We prove the existence and uniqueness of an equilibrium state with unit mass to the dissipative linear Boltzmann equation with hard-spheres collision kernel describing inelastic interactions of a gas particles with a fixed background. The equilibrium state is a universal Maxwellian distribution function with the same velocity as field particles and with a non-zero temperature lower than the background one. Moreover thanks to the $H$-Theorem we prove strong convergence of the solution to the Boltzmann equation towards the equilibrium.

**KEY WORDS:** Granular gases; equilibrium state; linear Fokker–Planck equation; trend to equilibrium.

## 1. INTRODUCTION

The present paper follows the very recent work[(25)] of Spiga and Toscani on the linear dissipative Boltzmann equation and generalizes a part of their results to the hard-spheres model.

In the last years, a significant interest has been devoted to the study of kinetic models for granular flows. The largest part of this work has its fundamentals on the *non-linear models* based upon generalizations of the Boltzmann–Enskog equation. We refer the reader to the review articles.[(9,16,17)] Most of the studies are dealing with inelastic Maxwell particles, both for the driven case[(4,7)] or for the free case.[(3)] Such (pseudo-) Maxwellian models enjoy nice mathematical simplifications and lead to exact analytical results[(12,13)] (see also the recent developments on the inelastic Kac model.[(23,24)]) However only a few papers are dealing with the inelastic hard-spheres models.[(5,15)]

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Despite their importance for practical applications, linear equations for dissipative models have been much less addressed. To our knowledge the only progresses on the matter are those of the afore-mentionned paper[25] and of Pettersson[2].[22] Such linear models arise due to suitable asymptotics on inelastic mixture models (corresponding to two species of different masses) describing typically the dynamics of fine polluting powders (particulate matter) interacting inelastically with a background gas (air). Namely, we are concerned with the time evolution of the distribution function $f(x,\mathbf{v},t)$ of particles of masses $m$ (representing the granular gas) colliding *inelastically* with particles with masses $m_1$ of a fixed background. Throughout this paper, the subscript (1) will be addressed to the fixed field particles whose distribution function is known and is assumed to be a normalized Maxwellian $M_1$ with given mass velocity and temperature. Note that, the grains being cohesionless, long-range interactions of any kind are irrelevant. Thus, the only model with real physical interest is the *hard-spheres model.* As first introduced in,[20] the evolution of $f(x,\mathbf{v},t)$ is given by

$$\frac{\partial f}{\partial t}(\mathbf{v},t)+\mathbf{v}\cdot\nabla_x f(x,\mathbf{v},t) = \frac{1}{2\pi\lambda}\int_{\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}\cdot\mathbf{n}|\left[\frac{1}{\epsilon^2}f(\mathbf{v}_\star)M_1(\mathbf{w}_\star)-f(\mathbf{v})M_1(\mathbf{w})\right]\mathrm{d}\mathbf{w}\ \mathrm{d}\mathbf{n}. \tag{1.1}$$

Here $\lambda$ denotes the constant mean free path, $\mathbf{q}$ is the relative velocity, $\mathbf{q}=\mathbf{v}-\mathbf{w}$. The velocities $(\mathbf{v}_\star,\mathbf{w}_\star)$ are the pre-collisional velocities of the so-called inverse collision, which results in $(\mathbf{v},\mathbf{w})$ as post-collisional velocities. The most important feature of the collision mechanism is its *inelastic character* which induces that (generally) it *does not preserve the total kinetic energy*. The constant parameter $0<\epsilon<1$ is called the restitution coefficient and measures the inelasticity of the collisions. Whenever $\epsilon=1$ we recover the usual linear Boltzmann equation (see Section 2 for details).

The main feature of this paper is to prove the existence and uniqueness of the (homogeneous) equilibrium state of the above Eq. (1.1). Precisely, we exhibit a (non-trivial) distribution function $f$ (depending on the velocity only) such that

$$\mathcal{Q}(f)=0,$$

[2]Let us also mention the related paper by Martin and Piasecki[20] which has been brought to our attention after the present one has been accepted for publication.

where $\mathcal{Q}$ denotes the right-hand side operator of (1.1). When $\epsilon = 1$, this question is trivial since the conservation of momentum and energy implies

$$\widetilde{M}_1(\mathbf{v})M_1(\mathbf{w}) = \widetilde{M}_1(\mathbf{v}_\star)M_1(\mathbf{w}_\star),$$

where $\widetilde{M}_1$ stands for the Maxwellian distribution with mass $m$ and same drift velocity and temperature as $M_1$. Then, one sees immediately that the integrand of $\mathcal{Q}$ vanishes for $f = \widetilde{M}_1$. Clearly, for $0 < \epsilon < 1$ this is no more the case and it appears impossible to determine so easily the (eventual) equilibrium state. Actually, the two following questions are far for being trivial:

- Does an equilibrium state exist?
- If it does, is it given by some suitable Maxwellian distribution?

In this paper, we answer positively to both questions by exhibiting an equilibrium state as a Maxwellian distribution with the same mass velocity as $M_1$ and with a universal non-zero temperature lower than the given background temperature. Moreover, this equilibrium state is unique, provided its mass is prescribed. Finally, this Maxwellian distribution coincide with the one derived in the pseudo-Maxwellian case[3].[(25)] Actually, it is also possible to show that, as in the *non-dissipative* case, the equilibrium state is universal in the sense that it does not depend on the collision kernel. Let us mention here that our results answer to some open questions from ref. 25 and complete the study of ref. 22 where the existence of an equilibrium state was used as an assumption for some of the results.

The two previous questions, as well as the problem of the rate of convergence towards equilibrium, have been recently addressed in the afore-mentioned paper[(25)] for the *pseudo-Maxwellian approximation*. This pseudo-Maxwellian approximation consists in replacing the relative velocity $\mathbf{q}$ appearing in the collision kernel $|\mathbf{q}\cdot\mathbf{n}|$ of $\mathcal{Q}$ by the unit vector in the direction of $\mathbf{q}$. The pseudo-Maxwellian model enjoys in particular two fundamental properties. First, the associated *moment equations are closed* with respect to the moments of the distribution function. Hence, it is possible to derive the time evolution of the drift velocity $\mathbf{u}(t)$ and the temperature $T(t)$ of $f(\mathbf{v},t)$ and to predict the mass velocity and temperature of the eventual equilibrium state. Moreover, as pointed out by Bobylev,[(2)] Maxwell models lend themselves to convenient Fourier analysis. These two important properties enabled to determine the Maxwellian equilibrium state

[3] Let us point out that this Maxwellian distribution has been first derived in ref. 20.

for the pseudo-Maxwellian model and to prove also exponential convergence of the solution to (1.1) towards the equilibrium (in the homogeneous setting).

Unfortunately, these two tools *do not apply* for the hard-spheres model (1.1) and we have to proceed in a different way. The main problem is actually to predict what should be the eventual steady-state. To do so, we derive *formally* a suitable linear Fokker–Planck equation associated to the dissipative Boltzmann model (1.1) (Section 3). The link between the two models is described through the asymptotics of the grazing collisions (see e.g., [8, Chapter II.9]) There is now a good amount of results on the matter for the elastic (non-linear) Boltzmann equation. We mention here the papers(10,11,27,28) on the connection between the non-linear Boltzmann equation and the Landau–Fokker–Planck equation and we refer to the works for linear problems. We emphasize the fact that our goal in this paper is absolutely not to prove rigorously any kind of asymptotics procedure. The results of Section 3 must only be viewed as a formal (but efficient) way to exhibit a suitable approximation of the collision operator $\mathcal{Q}$ (even if it is possible to make them rigorous, see appendix). We point out, that, to our knowledge, it is the first time (in kinetic theory) that such a limiting process is performed with this scope. The interest of this procedure is that the equilibrium state of the approximated Fokker–Planck operator is easy to obtain. It is actually a suitable Maxwellian distribution which appears then as the candidate for the stationary solution to (1.1). The main problem is then to prove that this Maxwellian is effectively a steady state for $\mathcal{Q}$. This will be done by means of a Fourier transform setting.

Let us explain now the organization of the paper. In Section 2, we describe briefly the dissipative Boltzmann linear model and its properties. In Section 3 we deal with the Fokker–Planck derivation of an approximation of $\mathcal{Q}$ which shall help us to recognize the nature of the equilibrium state. Then, in Section 4, we show that the Maxwellian obtained by the above procedure is really a stationary solution to (1.1). In Section 5 we prove thanks to the so-called $H$-Theorem that the equilibrium state is unique (provided its mass is prescribed) and that the solution to (1.1) converges (in the strong $L^1$-sense) towards the equilibrium. Finally, we end this paper by some open questions and perspectives.

## 2. THE DISSIPATIVE LINEAR BOLTZMANN EQUATION

As we told it in Introduction, we are concerned in this paper with the evolution of the distribution function $f(\mathbf{v}, t)$ of granular gas particles with masses $m$ which undergo inelastic collisions with the field particles (of masses $m_1$) of a fixed background. The background is supposed to be at thermodynamical equilibrium with given temperature $T_1$ and given mass velocity $\mathbf{u}_1$, i.e., its distribution function is the following *normalized Maxwellian*:

$$M_1(\mathbf{v}) = \left(\frac{m_1}{2\pi T_1}\right)^{3/2} \exp\left\{-\frac{m_1(\mathbf{v}-\mathbf{u}_1)^2}{2T_1}\right\} \quad \mathbf{v} \in \mathbb{R}^3.$$

The main feature of dissipative (inelastic) collisions is that part of the normal relative velocity is lost, that is

$$(\mathbf{v}^\star - \mathbf{w}^\star) \cdot \mathbf{n} = -\epsilon(\mathbf{v}-\mathbf{w}) \cdot \mathbf{n}, \tag{2.1}$$

where $\mathbf{n} \in \mathbb{S}^2$ is the unit vector in the direction of impact, $(\mathbf{v}, \mathbf{w})$ stand for the velocities before impact whereas $(\mathbf{v}^\star, \mathbf{w}^\star)$ denote the post-collisional velocities. The so-called (constant) *restitution coefficient* $\epsilon$ is such that $0 < \epsilon < 1$; the case $\epsilon = 1$ corresponding to elastic collision mechanism. Thanks to (2.1) and assuming the conservation of momentum

$$m\mathbf{v}^\star + m_1\mathbf{w}^\star = m\mathbf{v} + m_1\mathbf{w}$$

one finds the following collision mechanism

$$\begin{cases} \mathbf{v}^\star = \mathbf{v} - 2\alpha(1-\beta)[(\mathbf{v}-\mathbf{w}) \cdot \mathbf{n}]\mathbf{n} \\ \mathbf{w}^\star = \mathbf{w} + 2(1-\alpha)(1-\beta)[(\mathbf{v}-\mathbf{w}) \cdot \mathbf{n}]\mathbf{n}, \end{cases} \tag{2.2}$$

where $\alpha$ is the mass ratio and $\beta$ denotes the inelasticity parameter:

$$\alpha = \frac{m_1}{m+m_1}, \quad \beta = \frac{1-\epsilon}{2},$$

i.e., $0 < \alpha < 1$ (where we exclude the peculiarities of the limiting cases of Lorentz and Rayleigh gases) and $0 < \beta < 1/2$. We refer to ref.15 for a description of the geometry of the collisions. It is easy to see that system (2.2) is invertible and provides the pre-collisional velocities of the so-called inverse collisions, resulting in $(\mathbf{v}, \mathbf{w})$ as post-collisional velocities:

$$\begin{cases} \mathbf{v}_\star = \mathbf{v} - 2\alpha\dfrac{1-\beta}{1-2\beta}[(\mathbf{v}-\mathbf{w}) \cdot \mathbf{n}]\mathbf{n} \\ \mathbf{w}_\star = \mathbf{w} + 2(1-\alpha)\dfrac{1-\beta}{1-2\beta}[(\mathbf{v}-\mathbf{w}) \cdot \mathbf{n}]\mathbf{n}. \end{cases}$$

In contrast to the elastic case ($\epsilon = 1$), such a collision mechanism induces dissipation of the kinetic energy:

$$m|\mathbf{v}^\star|^2 + m_1|\mathbf{w}^\star|^2 - (m|\mathbf{v}|^2 + m_1|\mathbf{w}|^2) = -4\frac{mm_1}{m+m_1}\beta(1-\beta)|\mathbf{q}\cdot\mathbf{n}|^2 \leqslant 0.$$

In space homogeneous conditions, upon using $\lambda$ as a time scale, Eq. (1.1) can be re-written in the dimensionless form:

$$\frac{\partial f}{\partial t}(\mathbf{v}, t) = \frac{1}{2\pi}\int_{\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}\cdot\mathbf{n}|\left[\frac{1}{\epsilon^2} f(\mathbf{v}_\star)M_1(\mathbf{w}_\star) - f(\mathbf{v})M_1(\mathbf{w})\right]\mathrm{d}\mathbf{w}\ \mathrm{d}\mathbf{n}. \quad (2.3)$$

The factor $\epsilon^{-2}$ in the gain term above appears respectively for the Jacobian of the transformation $\mathrm{d}\mathbf{v}_\star \mathrm{d}\mathbf{w}_\star$ into $\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}$ and from the length of the cylinders $|\mathbf{q}^\star\cdot\mathbf{n}| = \epsilon|\mathbf{q}\cdot\mathbf{n}|$ (see[9] for details) Let us define the (dissipative) linear Boltzmann collision operator (acting only on the velocity space)

$$\mathcal{Q}(f) = \frac{1}{2\pi}\int_{\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}\cdot\mathbf{n}|\left[\frac{1}{\epsilon^2} f(\mathbf{v}_\star)M_1(\mathbf{w}_\star) - f(\mathbf{v})M_1(\mathbf{w})\right]\mathrm{d}\mathbf{w}\ \mathrm{d}\mathbf{n}. \quad (2.4)$$

Note that, performing the change of variables $\mathbf{n} \to -\mathbf{n}$ leads to the equivalent expression:

$$\mathcal{Q}(f) = \frac{1}{\pi}\int_{\mathbb{R}^3\times\mathbb{S}^2} H(\mathbf{q}\cdot\mathbf{n}/|\mathbf{q}|)\mathbf{q}\cdot\mathbf{n}\left[\frac{1}{\epsilon^2} f(\mathbf{v}_\star)M_1(\mathbf{w}_\star) - f(\mathbf{v})M_1(\mathbf{w})\right]\mathrm{d}\mathbf{w}\ \mathrm{d}\mathbf{n},$$

where $H(\cdot)$ is the Heavyside step function. We can also define the collision operator by its action on the *observables*. Precisely, for any regular test-function $\psi(\mathbf{v})$

$$\langle\psi, \mathcal{Q}(f)\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}\cdot\mathbf{n}| f(\mathbf{v})M_1(\mathbf{w})[\psi(\mathbf{v}^\star) - \psi(\mathbf{v})]\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n}. \quad (2.5)$$

Clearly, $\psi(\mathbf{v}) \equiv 1$ is a collision invariant (mass conservation) whereas, in contrast to the elastic case, $\psi(\mathbf{v}) = \mathbf{v}$ and $\psi(\mathbf{v}) = \mathbf{v}^2$ are not (dispersion of kinetic energy). Note that an important feature of the hard-spheres model is that (even in the elastic case) the moments equations of $\mathcal{Q}(f)$ are not closed with respect to the ones of $f$.

## 3. THE FOKKER–PLANCK APPROXIMATION

In this section, we perform a *formal* derivation of a suitable linear Fokker–Planck equation obtained from (1.1) through a kind of grazing collisions asymptotics. We point out that, our aim in this paper, *is not* to prove rigorously the convergence of the (re-scaled) dissipative Boltzmann operator $\mathcal{Q}$ towards the Fokker–Planck operator $\mathcal{Q}_{\mathrm{FP}}$ (3.9) below as the collisions become grazing. *The limiting process we perform here must only be seen as an efficient tool to predict the nature of the equilibrium state of* $\mathcal{Q}$ (if it exists). Nevertheless, the following approximation result can be made rigorous and this shall be done in the appendix. In this section, we will only stay at a formal level. Let us assume that all the collisions concentrate around

$$|\mathbf{q}\cdot\mathbf{n}|/|\mathbf{q}| \sim 0. \tag{3.1}$$

Consequently, according to (2.2) one has $|\mathbf{v}^\star-\mathbf{v}|\sim 0$ and, for any smooth function $\psi$, one can perform a Taylor expansion of $\psi(\mathbf{v}^\star)$ around $\mathbf{v}$ leading, at the second order, to:

$$\begin{aligned}\psi(\mathbf{v}^\star) &= \psi(\mathbf{v})+\nabla_{\mathbf{v}}\psi(\mathbf{v})\cdot(\mathbf{v}^\star-\mathbf{v})+\frac{1}{2}\mathbb{D}^2\psi(\mathbf{v})(\mathbf{v}^\star-\mathbf{v})\otimes(\mathbf{v}^\star-\mathbf{v})+\mathrm{o}(|\mathbf{v}^\star-\mathbf{v}|^2)\\ &= \psi(\mathbf{v})-2\alpha(1-\beta)(\mathbf{q}\cdot\mathbf{n})\nabla_{\mathbf{v}}\psi(\mathbf{v})\cdot\mathbf{n}+\frac{(2\alpha(1-\beta)\mathbf{q}\cdot\mathbf{n})^2}{2}\mathbb{D}^2\psi(\mathbf{v})\cdot\mathbf{n}\otimes\mathbf{n}\\ &\quad +\mathrm{o}(|\mathbf{q}\cdot\mathbf{n}/|\mathbf{q}||^2)\end{aligned} \tag{3.2}$$

where $\mathbb{D}^2\psi$ is the Hessian matrix of $\psi$. The $\mathrm{o}(|\mathbf{v}^\star-\mathbf{v}|^2)$ term will be neglected in the sequel. One clearly observes that the expansion (3.2) is similar to that obtained in the study of elastic collisions between particles of same masses.[19] We point out here that this property is strongly related to the fact that the loss of the relative velocity due to inelasticity only occurs in the direction parallel to the collision direction (a different situation would occur by taking into account dissipation of the tangential relative velocity). Precisely, one notes that (3.1) implies also $\mathbf{v}'\sim\mathbf{v}$ where $\mathbf{v}'$ is the post-collisional velocity in the elastic case. The only difference is that, in the "classical" theory, the multiplicative constant $2\alpha(1-\beta)$ is taken to be equal to $1/2$. Thus, staying at a formal level, *taking into account dissipative collisions between particles of unequal masses does not lead to supplementary difficulties.*

Let us consider a referential frame with the $x$-axis directed along $\mathbf{q}$. Then,

$$\begin{cases} \mathbf{n}=(\cos\theta, \sin\theta\cos\xi, \sin\theta\sin\xi) \quad 0\leqslant\theta\leqslant\pi/2,\ 0\leqslant\xi\leqslant 2\pi, \\ \cos\theta=\dfrac{|\mathbf{q}\cdot\mathbf{n}|}{|\mathbf{q}|} \quad \text{and} \quad \mathrm{d}\mathbf{n}=\sin\theta\,\mathrm{d}\theta\mathrm{d}\xi. \end{cases} \tag{3.3}$$

Assuming that that the collisions concentrate around $\theta\sim\pi/2$, we define

$$b_\delta(\theta)=\frac{2}{\pi}\chi_{[\pi/2-\delta,\pi/2]}(\theta) \quad (\delta>0).$$

and

$$I_\delta=\int_0^{\pi/2} b_\delta(\theta)\cos^3\theta\ \sin\theta\,\mathrm{d}\theta.$$

One sees that

$$I_\delta\sim\frac{\delta^4}{2\pi} \quad \text{as} \quad \delta\sim 0. \tag{3.4}$$

Consequently, let us define the associated collision kernel

$$B_\delta(\mathbf{q},\mathbf{n})=b_\delta(\theta)|\mathbf{q}\cdot\mathbf{n}|$$

and denote by $\mathcal{Q}_\delta$ the collision operator obtained by replacing $|\mathbf{q}\cdot\mathbf{n}|$ by $\delta^{-4}B_\delta(\mathbf{q},\mathbf{n})$ in (1.1).

**Remark 3.1.** Note that the introduction of the multiplicative factor $\delta^{-4}$ can be seen as a suitable *time-scaling* in (1.1) (see Appendix 6 for further details).

Our aim is to show that the (re-scaled) operator $\mathcal{Q}_\delta$ can be considered as a suitable Fokker–Planck collision operator. Precisely, let us fix $f\in L^1(\mathbb{R}^3)$ and a smooth test-function $\psi(\mathbf{v})$. Using (2.5) with $\mathcal{Q}_\delta$ leads to

$$\langle\psi,\ \mathcal{Q}_\delta(f)\rangle=\frac{1}{2\pi\delta^4}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n})f(\mathbf{v})M_1(\mathbf{w})\left[\psi(\mathbf{v}^\star)-\psi(\mathbf{v})\right]\mathrm{d}\mathbf{v}\ \mathrm{d}\mathbf{w}\ \mathrm{d}\mathbf{n}.$$

Now, inserting the expansion (3.2) in the above expression leads to the following second order approximation:

$$\begin{aligned}\langle\psi,\, \mathcal{Q}_\delta(f)\rangle &= \mathbf{J}_\delta^1+\mathbf{J}_\delta^2\\ &= -\frac{2\alpha(1-\beta)}{2\pi\delta^4}\\ &\quad\times\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n})(\mathbf{q}\cdot\mathbf{n})f(\mathbf{v})M_1(\mathbf{w})\nabla_{\mathbf{v}}\psi(\mathbf{v})\cdot\mathbf{n}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}\\ &\quad+\frac{(2\alpha(1-\beta))^2}{4\pi\delta^4}\\ &\quad\times\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n})\,(\mathbf{q}\cdot\mathbf{n})^2\, f(\mathbf{v})M_1(\mathbf{w})\mathbb{D}^2\psi(\mathbf{v})\\ &\qquad\cdot\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}. \end{aligned}\tag{3.5}$$

To estimate $\mathbf{J}_\delta^1$, we first compute the integral with respect to d$\mathbf{n}$. According to (3.3)

$$\begin{aligned}\int_{\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n})(\mathbf{q}\cdot\mathbf{n})\mathbf{n}\mathrm{d}\mathbf{n} &= 2\pi|\mathbf{q}|\int_0^{\pi/2} B_\delta(\mathbf{q},\mathbf{n})(\cos^2\theta,0,0)\sin\theta\,\mathrm{d}\theta\\ &= 2\pi|\mathbf{q}|^2\int_0^{\pi/2} b_\delta(\theta)(\cos^3\theta,0,0)\sin\theta\,\mathrm{d}\theta\\ &= 2\pi|\mathbf{q}|^2(I_\delta,0,0)=2\pi I_\delta|\mathbf{q}|\mathbf{q}.\end{aligned}$$

Therefore

$$\mathbf{J}_\delta^1=-I_\delta\frac{2\alpha(1-\beta)}{\delta^4}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\mathbf{q}|^2 f(\mathbf{v})M_1(\mathbf{w})\nabla_{\mathbf{v}}\psi(\mathbf{v},t)\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}$$

One notes, because of (3.4), that the coefficient in front of the above integral goes to $-\alpha(1-\beta)/\pi$ as $\delta$ goes to 0. Consequently

$$\mathbf{J}_\delta^1\simeq-\frac{\alpha(1-\beta)}{\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\mathbf{q}|^2 f(\mathbf{v})M_1(\mathbf{w})\nabla_{\mathbf{v}}\psi(\mathbf{v},t)\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\,\mathrm{d}\mathbf{v}\ \mathrm{d}\mathbf{w}.\tag{3.6}$$

We proceed in the same way for $\mathbf{J}_\delta^2$. One has first

$$\int_{\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n})\,(\mathbf{q}\cdot\mathbf{n})^2\,\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\mathbf{n} = |\mathbf{q}|^3 \int_0^{\pi/2} b_\delta(\theta)\cos^3\theta\,\sin\theta\,\mathrm{d}\theta \int_0^{2\pi} n\otimes n\,\mathrm{d}\xi$$
$$= 2\pi|\mathbf{q}|^3 \int_0^{\pi/2} b_\delta(\theta)\cos^3\theta\,\sin\theta$$
$$\cdot\,\mathbf{Diag}\left[\cos^2\theta,\,\frac{1}{2}\sin^2\theta,\,\frac{1}{2}\sin^2\theta\right]\mathrm{d}\theta, \quad (3.7)$$

where $\mathbf{Diag}[a_1,a_2,a_3]$ is the diagonal matrix in $\mathbb{R}^3\times\mathbb{R}^3$ whose diagonal entries are $a_i$ $(i=1,2,3)$. Now, defining

$$K_\delta = \int_0^{\pi/2} b_\delta(\theta)\cos^5\theta\,\sin\theta\,\mathrm{d}\theta$$

one gets

$$\int_{\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n})\,(\mathbf{q}\cdot\mathbf{n})^2\,\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\mathbf{n} = 2\pi|\mathbf{q}|^3\mathbf{Diag}\left[K_\delta,\,\frac{1}{2}(I_\delta-K_\delta),\,\frac{1}{2}(I_\delta-K_\delta)\right]$$

and

$$\mathbf{J}_\delta^2 = \frac{(2\alpha(1-\beta))^2}{2\delta^4}\int_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{q}|^3 f(\mathbf{v})M_1(\mathbf{w})\mathbb{D}^2\psi(\mathbf{v},t)$$
$$\cdot\,\mathbf{Diag}\left[K_\delta,\,\frac{1}{2}(I_\delta-K_\delta),\,\frac{1}{2}(I_\delta-K_\delta)\right]\mathrm{d}\mathbf{v}\;\mathrm{d}\mathbf{w}.$$

Now, since $K_\delta$ is negligible with respect to $I_\delta$ one gets the following approximation

$$\mathbf{J}_\delta^2 \simeq \frac{\alpha^2(1-\beta)^2}{2\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{q}|^3 f(\mathbf{v})M_1(\mathbf{w})\mathbb{D}^2\psi(\mathbf{v})\cdot\mathbf{Diag}[0,1,1]\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}. \quad (3.8)$$

Let for any $\mathbf{z}\in\mathbb{R}^3$ ($\mathbf{z}\neq 0$), $\mathbb{S}(\mathbf{z})$ be the symmetric *matrix*

$$\mathbb{S}(\mathbf{z})=\mathrm{Id}-\frac{\mathbf{z}\otimes\mathbf{z}}{|\mathbf{z}|^2},$$

i.e., $\mathbb{S}(\mathbf{z})$ is the projection on the space orthogonal to $\mathbf{z}$. Then, combining (3.5), (3.6) and (3.8), one obtains the following approximation for $\mathcal{Q}_\delta$:

$$\langle\psi,\ \mathcal{Q}_\delta(f)\rangle \simeq -\frac{\alpha(1-\beta)}{\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\mathbf{q}|^2 f(\mathbf{v})M_1(\mathbf{w})\nabla_{\mathbf{v}}\psi(\mathbf{v})\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}$$
$$+\frac{\alpha^2(1-\beta)^2}{2\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\mathbf{q}|^3 f(\mathbf{v})M_1(\mathbf{w})\mathbb{D}^2\psi(\mathbf{v})\cdot\mathbb{S}(\mathbf{v}-\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}.$$

Now, straightforward computations, using the fact that $2|\mathbf{q}|\mathbf{q}=\mathbf{Div}_{\mathbf{v}}(|\mathbf{v}-\mathbf{w}|^3\mathbf{S}(\mathbf{v}-\mathbf{w}))$, yield

$$\langle\psi,\ \mathcal{Q}_\delta(f)\rangle \simeq -\frac{1}{2\pi}\int_{\mathbb{R}^3}\mathrm{d}\mathbf{v}\,\nabla_{\mathbf{v}}\psi(\mathbf{v})\cdot\int_{\mathbb{R}^3}|\mathbf{v}-\mathbf{w}|^3\mathbb{S}(\mathbf{v}-\mathbf{w})\,\{\kappa M_1(\mathbf{w})\nabla_{\mathbf{v}}f(\mathbf{v})$$
$$+(\kappa-\mu)f(\mathbf{v})\nabla_{\mathbf{w}}M_1(\mathbf{w})\}\,\mathrm{d}\mathbf{w}$$

where we introduced the following parameters

$$\kappa=\alpha^2(1-\beta)^2 \quad\text{and}\quad \mu=\alpha(1-\beta).$$

Since the above approximation is valid for arbitrary smooth function $\psi$, one sees that, as $\delta$ goes to 0, $\mathcal{Q}_\delta$ can be approximated by the following Fokker–Planck operator (up to the constant $1/2\pi$)

$$\mathcal{Q}_{\mathrm{FP}}(g)(\mathbf{v}) = \nabla_{\mathbf{v}}\cdot\int_{\mathbb{R}^3}|\mathbf{v}-\mathbf{w}|^3\mathbb{S}(\mathbf{v}-\mathbf{w})\cdot\{\kappa M_1(\mathbf{w})\nabla_{\mathbf{v}}g(\mathbf{v})$$
$$+(\kappa-\mu)g(\mathbf{v})\nabla_{\mathbf{w}}M_1(\mathbf{w})\}\mathrm{d}\mathbf{w}.$$

Let us write $\mathcal{Q}_{\mathrm{FP}}$ in a nicer way: using the fact that $\mathbb{S}(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w})=0$ and that

$$\nabla_{\mathbf{w}}M_1(\mathbf{w})=-\frac{m_1(\mathbf{w}-\mathbf{u}_1)}{T_1}M_1(\mathbf{w}),$$

one has

$$\mathcal{Q}_{\mathrm{FP}}(g)(\mathbf{v}) = \kappa \nabla_{\mathbf{v}} \cdot \int_{\mathbb{R}^3} |\mathbf{v}-\mathbf{w}|^3 M_1(\mathbf{w}) \mathbb{S}(\mathbf{v}-\mathbf{w}) \cdot \left\{ \nabla_{\mathbf{v}} g(\mathbf{v}) - \frac{m_1(\kappa-\mu)}{\kappa T_1}(\mathbf{v}-\mathbf{u}_1) g(\mathbf{v}) \right\} \mathrm{d}\mathbf{w}$$
$$= \kappa \nabla_{\mathbf{v}} \cdot \left[ \mathbb{A}(\mathbf{v}) \cdot \left\{ \nabla_{\mathbf{v}} g(\mathbf{v}) + \frac{m_1(\mu-\kappa)}{\kappa T_1}(\mathbf{v}-\mathbf{u}_1) g(\mathbf{v}) \right\} \right], \quad (3.9)$$

where $\mathbb{A}(\mathbf{v})$ denotes the invertible matrix

$$\mathbb{A}(\mathbf{v}) = \int_{\mathbb{R}^3} |\mathbf{v}-\mathbf{w}|^3 M_1(\mathbf{w}) \mathbb{S}(\mathbf{v}-\mathbf{w}) \, \mathrm{d}\mathbf{w}.$$

**Remark 3.2.** In a different spirit, Brey *et al.*[6] derived a linear Fokker–Planck equation from (1.1) in the limit of small mass ratio ($\alpha \to 0$). We also point out that it is possible to consider (see e.g. ref. 26) the quasi-elastic approximation of $\mathcal{Q}$ assuming that $\beta \ll 1$. We adopt here the grazing collisions asymptotics since it preserves the parameters $\alpha$ and $\beta$ (and therefore the inelasticity) modifying only the geometry of the collisions. Note however that the existence of an equilibrium state for the linear quasi-elastic approximation of $\mathcal{Q}$ is an open problem to our knowledge.

Now, one sees immediately that the above procedure preserves the equilibrium state. Precisely, if $F$ is an equilibrium state of $\mathcal{Q}_\delta$, then $\mathcal{Q}_{\mathrm{FP}}(F) = 0$. This strongly suggest that one has to select the candidate for being an equilibrium state of $\mathcal{Q}$ as the one of $\mathcal{Q}_{\mathrm{FP}}$. Of course, the interest of the above procedure lies in the fact that this latter is easy to exhibit. Indeed, it is obvious from (3.9) that the unique solution with unit mass to $\mathcal{Q}_{\mathrm{FP}}(g) = 0$ is given by a Maxwellian distribution with drift velocity $\mathbf{u}_1$ and temperature

$$\frac{T^{\#}}{m} = \left\{ \frac{m_1(\mu-\kappa)}{\kappa T_1} \right\}^{-1}.$$

This suggests the following dichotomy:

- Either $\mathcal{Q}(f) = 0$ has no non-trivial solution.

- Either the unique solution to $\mathcal{Q}(f)=0$ with unit mass is the Maxwellian:

$$M(\mathbf{v})=\left(\frac{m}{2\pi T^{\#}}\right)^{3/2}\exp\left\{-\frac{m(\mathbf{v}-\mathbf{u}_1)^2}{2T^{\#}}\right\}\quad \mathbf{v}\in\mathbb{R}^3, \tag{3.10}$$

where

$$T^{\#}=\frac{(1-\alpha)(1-\beta)}{1-\alpha(1-\beta)}T_1. \tag{3.11}$$

At this point, we remark that the above Maxwellian distribution is exactly the equilibrium state found in ref. 25 in their study of the pseudo-Maxwellian approximation of (1.1). This supports our belief that $M$ is indeed the steady state of $\mathcal{Q}$ and that, moreover, it is also a *universal* stationary solution (independent of the collision kernel) as it occurs in the elastic case. This will be proved rigorously in the following section.

## 4. IS THE MAXWELLIAN THE EQUILIBRIUM STATE?

The problem of finding the equilibrium state of the linear Boltzmann equation (1.1) has now been reduced to determine whether $\mathcal{Q}(M)\equiv 0$ or not, where

$$M(\mathbf{v})=\left(\frac{m}{2\pi T^{\#}}\right)^{3/2}\exp\left\{-\frac{m(\mathbf{v}-\mathbf{u}_1)^2}{2T^{\#}}\right\}\quad \forall\mathbf{v}\in\mathbb{R}^3,$$

with $T^{\#}=[(1-\alpha)(1-\beta)/1-\alpha(1-\beta)]T_1$. Surprisingly, apart for the peculiar case of 1D-model, we have not been able to prove by direct computation that

$$\mathcal{Q}(M)(\mathbf{v})=0\quad \mathbf{v}\in\mathbb{R}^3.$$

Actually, we prove this result through Fourier analysis, i.e., we show that

$$\widehat{\mathcal{Q}(M)}(\xi)=0\qquad \text{for any } \xi\in\mathbb{R}^3,$$

where $\widehat{\mathcal{Q}(M)}(\xi)$ denotes the Fourier transform of $\mathcal{Q}(M)$. By (2.5), it is given by

$$\widehat{\mathcal{Q}(M)}(\xi)=\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}|\mathbf{q}\cdot\mathbf{n}|M(\mathbf{v})M_1(\mathbf{w})[\exp\{-i\xi\cdot\mathbf{v}^\star\}-\exp\{-i\xi\cdot\mathbf{v}\}]\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}.$$

One sees immediately that, up to a translation of the referential frame, one can assume that

$$\mathbf{u}_1 = 0.$$

For the sake of simplicity, let us introduce the following parameter

$$C = \left( \frac{m\, m_1}{2T_1\, T^{\#}} \right)^{3/2}$$

and recall that $\mu = \alpha(1-\beta)$. Then,

$$\widehat{\mathcal{Q}(M)}(\xi) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\mathbf{q}| M(\mathbf{v}) M_1(\mathbf{w}) \exp\{-i\xi \cdot \mathbf{v}\} d\mathbf{v}\, d\mathbf{w} \times \int_{\mathbb{S}^2} \left| \frac{\mathbf{q}}{|\mathbf{q}|} \cdot \mathbf{n} \right| \left( \exp\{2\, i\, \mu\, (\mathbf{q} \cdot \mathbf{n})\, (\xi \cdot \mathbf{n})\} - 1 \right) d\mathbf{n}.$$

Now, the *key point of our computations* is the identity

$$\begin{aligned} M(\mathbf{v}) M_1(\mathbf{w}) &= C \exp\left\{ -\frac{m_1}{2\mu T_1} \left[ \mu \mathbf{w}^2 + (1-\mu) \mathbf{v}^2 \right] \right\} \\ &= C \exp\left\{ -\frac{m_1}{2\mu T_1} \left[ \mu(1-\mu) \mathbf{q}^2 + (\mathbf{v} - \mu \mathbf{q})^2 \right] \right\}, \end{aligned}$$

where we used the fact that $\frac{m}{T^{\#}} = -(\mu - 1) \frac{m_1}{\mu T_1}$. Then,

$$\widehat{\mathcal{Q}(M)}(\xi) = C \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\mathbf{q}| \exp\left\{ -\frac{m_1}{2\mu T_1} \left[ \mu(1-\mu) \mathbf{q}^2 + (\mathbf{v} - \mu \mathbf{q})^2 \right] \right\} \exp\{-i\xi \cdot \mathbf{v}\} d\mathbf{v}\, d\mathbf{w} \times \int_{\mathbb{S}^2} \left| \frac{\mathbf{q}}{|\mathbf{q}|} \cdot \mathbf{n} \right| \left( \exp\{2\, i\, \mu\, (\mathbf{q} \cdot \mathbf{n})\, (\xi \cdot \mathbf{n})\} - 1 \right) d\mathbf{n}.$$

The change of variables $(\mathbf{v}, \mathbf{w}) \to (\mathbf{v}, \mathbf{q})$ yields

$$\begin{aligned}\widehat{\mathcal{Q}(M)}(\xi) = \; & C \int_{\mathbb{R}^3} |\mathbf{q}| \exp\left\{-\frac{m_1}{2\mu T_1}\mu(1-\mu)\mathbf{q}^2\right\} \mathrm{d}\mathbf{q} \\ & \times \int_{\mathbb{R}^3} \exp\left\{-\frac{m_1}{2\mu T_1}(\mathbf{v}-\mu\mathbf{q})^2\right\} \exp\{-i\xi \cdot \mathbf{v}\}\mathrm{d}\mathbf{v} \\ & \times \int_{\mathbb{S}^2} |\frac{\mathbf{q}}{|\mathbf{q}|} \cdot \mathbf{n}| \left( \exp\{2\, i\, \mu\, (\mathbf{q}\cdot\mathbf{n})\, (\xi\cdot\mathbf{n})\} - 1 \right) \mathrm{d}\mathbf{n}.\end{aligned}$$

Performing first the second integral leads to

$$\begin{aligned}\widehat{\mathcal{Q}(M)}(\xi) = C \exp\left\{-\frac{\mu T_1}{2m_1}\xi^2\right\} \int_{\mathbb{R}^3} |\mathbf{q}| \exp\left\{-\frac{m_1}{2T_1}(1-\mu)\mathbf{q}^2\right\} \exp\{-i\mu\mathbf{q}\cdot\xi\}\mathrm{d}\mathbf{q} \\ \times \int_{\mathbb{S}^2} |\frac{\mathbf{q}}{|\mathbf{q}|} \cdot \mathbf{n}| \left( \exp\{2\, i\, \mu\, (\mathbf{q}\cdot\mathbf{n})\, (\xi\cdot\mathbf{n})\} - 1 \right) \mathrm{d}\mathbf{n}\end{aligned}$$

where we used the fact that the Fourier transform of the Gaussian $\exp\{-(\mathbf{v}-\mathbf{u})^2/2\Theta\}$ is equal to $C_\Theta \exp\{-i\mathbf{u}\cdot\xi - \Theta/2\xi^2\}$ for any $\Theta > 0$ and $\mathbf{u} \in \mathbb{R}^3$ (here $\mathbf{u} = \mu\mathbf{q}$ and $\Theta = m_1/\mu T_1$) where $C_\Theta$ is a multiplicative constant depending only on $\Theta$. Now, as pointed out first by Bobylev, (2) the inner integral on the unit sphere is an isotropic function of the vectors $\xi$ and $\mathbf{q}$ and is therefore equal to

$$\int_{\mathbb{S}^2} |\xi\cdot\mathbf{n}/|\xi||\, \left( \exp\{2\, i\, \mu\, (\mathbf{q}\cdot\mathbf{n})\, (\xi\cdot\mathbf{n})\} - 1 \right) \mathrm{d}\mathbf{n}.$$

Consequently,

$$\begin{aligned}\widehat{\mathcal{Q}(M)}(\xi) = \; & C \exp\left\{-\frac{\mu T_1}{2m_1}\xi^2\right\} \int_{\mathbb{R}^3} |\mathbf{q}| \exp\left\{-\frac{m_1}{2T_1}(1-\mu)\mathbf{q}^2\right\} \mathrm{d}\mathbf{q} \\ & \times \int_{\mathbb{S}^2} |\xi\cdot\mathbf{n}/|\xi||\, \Big( \exp\{-i\, \mu\, (\mathbf{q}\cdot\xi^+)\} - \exp\{-i\mu\mathbf{q}\cdot\xi\} \Big) \mathrm{d}\mathbf{n},\end{aligned}$$

where

$$\xi^+ = \xi - 2(\xi\cdot\mathbf{n})\mathbf{n}.$$

Now, since the last integral on the unit sphere only depends on $|\xi|$ and $\xi \cdot \mathbf{q}$, this proves that

$$\widehat{\mathcal{Q}(M)}(\xi) = 0 \qquad \text{for any } \xi \in \mathbb{R}^3,$$

because $|\xi^+| = |\xi|$. We proved the existence result.

**Theorem 4.1.** The Maxwellian distribution

$$M(\mathbf{v}) = \left(\frac{m}{2\pi T^{\#}}\right)^{3/2} \exp\left\{-\frac{m(\mathbf{v}-\mathbf{u}_1)^2}{2T^{\#}}\right\} \quad \mathbf{v} \in \mathbb{R}^3,$$

with $T^{\#} = [(1-\alpha)(1-\beta)/1-\alpha(1-\beta)]T_1$ is an equilibrium state for $\mathcal{Q}$.

**Remark 4.2.** (*Universality of the Maxwellian*) One can easily generalize the above computations to show that the Maxwellian (3.10) is an equilibrium state of any collision operator $\mathcal{Q}_B$ enjoying the following weak form

$$\langle \psi, \mathcal{Q}_B(f) \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B(\mathbf{q}, \mathbf{n}) f(\mathbf{v}) M_1(\mathbf{w}) [\psi(\mathbf{v}^{\star}) - \psi(\mathbf{v})] \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n} \tag{4.1}$$

for any smooth function $\psi$. The collision kernel $B(\cdot,\cdot)$ is given by

$$B(\mathbf{q}, \mathbf{n}) = |\mathbf{q}|^{\gamma} b(\mathbf{q} \cdot \mathbf{n}/|\mathbf{q}|) \tag{4.2}$$

with $-1 \leqslant \gamma \leqslant 1$ and $b(\cdot)$ non-negative. This shows that, as it happens in the elastic case, *the equilibrium state of the dissipative (linear) Boltzmann equation is universal* in the sense that it does not depend on the collision kernel.

**Remark 4.3.** Note that for a general collision kernel $B(\mathbf{q}, \mathbf{n})$ it is convenient to use (4.1) as a definition for $\mathcal{Q}_B$ instead of its strong form:

$$\mathcal{Q}_B(f) = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{S}^2} B(\mathbf{q}, \mathbf{n}) \{ J(\mathbf{q}, \mathbf{n}) f(\mathbf{v}_{\star}) M_1(\mathbf{w}_{\star}) - f(\mathbf{v}) M_1(\mathbf{w}) \} \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n},$$

where the factor $J$ depends of $B$, $\alpha$ and $\beta$ in a complicated way.

## 5. ON THE TREND TO EQUILIBRIUM

In this section we investigated the large-time behavior of the solution to the linear dissipative Boltzmann Eq. (1.1). Precisely, let $f_0$ be a given (non-negative) distribution function and consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial f}{\partial t}(\mathbf{v},t) = \mathcal{Q}(f)(\mathbf{v},t) & \mathbf{v}\in\mathbb{R}^3, \quad t\geqslant 0 \\ f(\mathbf{v},t=0) = f_0(\mathbf{v}) \end{cases} \tag{5.1}$$

Since the above problem is linear (and homogeneous), it is not difficult to construct a (non-negative) mild solution to (5.1) by a simple iterative method. Moreover, this solution is unique and mass is preserved:

$$\int_{\mathbb{R}^3} f(\mathbf{v},t)\mathrm{d}\mathbf{v} = \int_{\mathbb{R}^3} f_0(\mathbf{v})\mathrm{d}\mathbf{v} \qquad \text{for any } t\geqslant 0.$$

For further details, we refer the reader to ref. 22 where a more general framework is taken into account (covering in particular inhomogeneous equation with suitable boundary conditions).

A fundamental task in kinetic theory is to determine whether the solution to (5.1) converges toward the equilibrium state of $\mathcal{Q}$ or not. Such a result has been proved by Petterson[(22)] for collision kernels of the form (4.2) with $-1<\gamma<1$ (corresponding to hard or soft interactions).

Actually, to get such a result one has first to prove that the equilibrium state we exhibited is unique. This can be done thanks to the $H$-Theorem.

**On the H-Theorem.** Let us recall here the linear $H$-Theorem for the dissipative Boltzmann Eq. (2.3) This result has been first established by Pettersson,[(22)] assuming the existence of an equilibrium state for $\mathcal{Q}$. We point out that, in contrast to what happens in the non-linear setting, the existence of such a steady-state is *necessary* to prove the linear $H$-Theorem.

Now that this existence result is established, we are able to state the corresponding $H$-Theorem and this shall enable us also to prove the uniqueness of the stationary solution.

We give here an elementary formal proof of the $H$-Theorem (for details see Pettersson[(22)] for a more general result for the linear inhomogeneous equation with suitable boundary conditions). Let $\Phi:\mathbb{R}^+\to\mathbb{R}$ be a *convex* $C^1$-function.

Define the associated entropy functional

$$H_\Phi(f|M) = \int_{\mathbb{R}^3} M(\mathbf{v})\Phi\left(\frac{f(\mathbf{v})}{M(\mathbf{v})}\right)\mathrm{d}\mathbf{v}, \tag{5.2}$$

where $M(\mathbf{v})$ is the Maxwellian (3.10). The $H$-Theorem asserts that $H_\Phi(\cdot|M)$ is a Lyapounov functional for the linear Boltzmann equation.

**Theorem 5.1.** (*H-Theorem*) Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a convex $C^1$-function and let $f_0$ be a distribution function with unit mass such that $H_\Phi(f_0|M) < \infty$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} H_\Phi(f(t)|M) \leqslant 0 \quad (t \geqslant 0)$$

where $f(t)$ stands for the (unique) solution to (5.1).

*Proof.* It is clear that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H_\Phi(f(t)|M) &= \int_{\mathbb{R}^3} \frac{\partial f}{\partial t}(\mathbf{v}, t)\Phi'\left(\frac{f(\mathbf{v}, t)}{M(\mathbf{v})}\right)\mathrm{d}\mathbf{v} \\ &= \int_{\mathbb{R}^3} \mathcal{Q}(f)(\mathbf{v}, t)\Phi'\left(\frac{f(\mathbf{v}, t)}{M(\mathbf{v})}\right)\mathrm{d}\mathbf{v}\end{aligned}$$

and this amounts to show that

$$\int_{\mathbb{R}^3} \mathcal{Q}(f)(\mathbf{v})\Phi'\left(\frac{f(\mathbf{v})}{M(\mathbf{v})}\right)\mathrm{d}\mathbf{v} \leqslant 0 \tag{5.3}$$

for any distribution function $f$ with unit mass for which the above integral is meaningful. From (2.5)

$$\begin{aligned}\int_{\mathbb{R}^3} \mathcal{Q}(f)(\mathbf{v})\Phi'\left(\frac{f(\mathbf{v})}{M(\mathbf{v})}\right)\mathrm{d}\mathbf{v} = \frac{1}{2\pi} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} &|\mathbf{q}\cdot\mathbf{n}| f(\mathbf{v}) M_1(\mathbf{w}) \\ &\times \left\{\Phi'\left(\frac{f(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right) - \Phi'\left(\frac{f(\mathbf{v})}{M(\mathbf{v})}\right)\right\}\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}\end{aligned}$$

and this last integral is also equal to

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}\cdot\mathbf{n}|M(\mathbf{v})M_1(\mathbf{w}) \left\{\left[\frac{f(\mathbf{v})}{M(\mathbf{v})}-\frac{f(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right]\Phi'\left(\frac{f(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right)\right.$$
$$+\frac{f(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\Phi'\left(\frac{f(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right)$$
$$\left.-\frac{f(\mathbf{v})}{M(\mathbf{v})}\Phi'\left(\frac{f(\mathbf{v})}{M(\mathbf{v})}\right)\right\}\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}.$$

Now, since $\left\langle\frac{f}{M}\Phi'\left(\frac{f}{M}\right), \mathcal{Q}(M)\right\rangle=0,$

$$\int_{\mathbb{R}^3}\mathcal{Q}(f)(\mathbf{v})\Phi'\left(\frac{f(\mathbf{v})}{M(\mathbf{v})}\right)\mathrm{d}\mathbf{v}= \frac{1}{2\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}|\mathbf{q}\cdot\mathbf{n}|M(\mathbf{v})M_1(\mathbf{w})$$
$$\times\left\{\frac{f(\mathbf{v})}{M(\mathbf{v})}-\frac{f(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right\}$$
$$\times\Phi'\left(\frac{f(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right)\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}. \tag{5.4}$$

The conclusion follows since, $\Phi$ being convex,

$$\Phi'(a)(b-a)\leqslant\Phi(b)-\Phi(a) \quad (a,\, b\in\mathbb{R})$$

and $\left\langle\Phi\left(\frac{f}{M}\right), \mathcal{Q}(M)\right\rangle=0.$ ∎

As a consequence of the $H$-Theorem, one has immediately the following uniqueness result (due to Pettersson[22] in a more general setting).

**Corollary 5.2.** The Maxwellian distribution $M$ given by (3.10) is the unique stationary solution to (1.1) with unit mass.

*Proof.* Let $F$ be another equilibrium state with unit mass. Then, according to (5.4) with $\Phi(z)=\frac{(z-1)^2}{2}$ one sees that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}|\mathbf{q}\cdot\mathbf{n}|\,M(\mathbf{v})M_1(\mathbf{w})\left\{\frac{F(\mathbf{v})}{M(\mathbf{v})}-\frac{F(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right\}\frac{F(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}=0.$$

This implies that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}\cdot\mathbf{n}|M(\mathbf{v})M_1(\mathbf{w})\left\{\frac{F(\mathbf{v})}{M(\mathbf{v})}-\frac{F(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right\}^2 \mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}$$
$$=\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}\cdot\mathbf{n}|F(\mathbf{v})M_1(\mathbf{w})\left\{\frac{F(\mathbf{v})}{M(\mathbf{v})}-\frac{F(\mathbf{v}^\star)}{M(\mathbf{v}^\star)}\right\}\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}=0,$$

where this last integral is null since $\mathcal{Q}(F)=0$. Consequently, one gets that

$$\frac{F(\mathbf{v})}{M(\mathbf{v})}=\frac{F(\mathbf{v}^\star)}{M(\mathbf{v}^\star)} \qquad \text{for any } (\mathbf{v},\mathbf{w})\in\mathbb{R}^3$$

and this last identity leads, as in the elastic case, to $F=M$. ∎

**The evolution of some second moment.** Now, to prove that the solution to the Cauchy problem (5.1) converges towards the (unique) equilibrium state, one has to establish some suitable *a priori* estimates. Actually, in contrast to the elastic case and because of the lack of collision invariants, it is not trivial to estimate the evolution of the moments of $f(\mathbf{v},t)$. This difficulty is peculiar to the hard-spheres model and does not occurs for pseudo-Maxwellian molecules.[25] For long-range interactions forces, Pettersson proves uniform estimates on the higher moments of $f(\mathbf{v},t)$ (see ref. 22, Theorem 4.1). Unfortunately, his arguments do not apply for the hard-sphere model. Nevertheless, it is possible to show that some second moment of the solution to (5.1) remains bounded. Precisely, let the initial distribution function $f_0\in L^1(\mathbb{R}^3)$ have *unit mass* and let $f(t)$ be the solution to the Cauchy problem (5.1). Recall that for any $t\geqslant 0$, $f(\mathbf{v},t)$ has also unit mass. One defines the following second moment of $f(t)$:

$$T(t)=\frac{m}{3}\int_{\mathbb{R}^3} f(\mathbf{v},t)(\mathbf{v}-\mathbf{u}_1)^2\mathrm{d}\mathbf{v}.$$

Note that $T(t)$ is not *stricto sensu* the temperature of $f(t)$ which is defined by replacing the above velocity $\mathbf{u}_1$ by the drift velocity of $f(t)$. Define also

$$F(t)=\int_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{v}-\mathbf{w}|^2 f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}.$$

One has

$$F(t)=\int_{\mathbb{R}^3}(\mathbf{v}-\mathbf{u}_1)^2 f(\mathbf{v},t)\int_{\mathbb{R}^3}M_1(\mathbf{w})\mathrm{d}\mathbf{w}+\int_{\mathbb{R}^3}f(\mathbf{v},t)\mathrm{d}\mathbf{v}\int_{\mathbb{R}^3}(\mathbf{w}-\mathbf{u}_1)^2 M_1(\mathbf{w})\,\mathrm{d}\mathbf{w}$$
$$-2\int_{\mathbb{R}^3}(v-\mathbf{u}_1)f(\mathbf{v},t)\mathrm{d}\mathbf{v}\cdot\int_{\mathbb{R}^3}(\mathbf{w}-\mathbf{u}_1)M_1(\mathbf{w})\mathrm{d}\mathbf{w}$$

and this last integral equals to zero by definition of $\mathbf{u}_1$. Therefore

$$F(t)=\frac{3}{m}T(t)+\frac{3}{m_1}T_1. \tag{5.5}$$

Now, from (2.5), one has

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t}=\frac{m}{6\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}|\mathbf{q}\cdot\mathbf{n}|f(\mathbf{v},t)M_1(\mathbf{w})\left\{(\mathbf{v}^\star-\mathbf{u}_1)^2-(\mathbf{v}-\mathbf{u}_1)^2\right\}\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}$$

and

$$\begin{aligned}(\mathbf{v}^\star-\mathbf{u}_1)^2-(\mathbf{v}-\mathbf{u}_1)^2 &= 4\alpha^2(1-\beta)^2|\mathbf{q}\cdot\mathbf{n}|^2-4\alpha(1-\beta)(\mathbf{q}\cdot\mathbf{n})(v-\mathbf{u}_1)\cdot\mathbf{n}\\ &= -4\alpha(1-\beta)[1-\alpha(1-\beta)]\,|\mathbf{q}\cdot\mathbf{n}|^2\\ &\quad+4\alpha(1-\beta)(\mathbf{q}\cdot\mathbf{n})(\mathbf{w}-\mathbf{u}_1)\cdot\mathbf{n}.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\mathrm{d}T(t)}{\mathrm{d}t}= &-\frac{2m}{3\pi}\mu(1-\alpha(1-\beta))\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}|\mathbf{q}\cdot\mathbf{n}|^3 f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}\\ &+\frac{2m}{3\pi}\mu\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}|\mathbf{q}\cdot\mathbf{n}|(\mathbf{q}\cdot\mathbf{n})((\mathbf{w}-\mathbf{u}_1)\cdot\mathbf{n})f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,d\mathbf{n}\end{aligned}$$

where we recall that $0<\mu=\alpha(1-\beta)<1$. Since

$$\int_{\mathbb{S}^2}|\mathbf{q}\cdot\mathbf{n}|^3\mathrm{d}\mathbf{n}=\pi|\mathbf{q}|^3 \quad \text{and} \quad \int_{\mathbb{S}^2}|\mathbf{q}\cdot\mathbf{n}|(\mathbf{q}\cdot\mathbf{n})(\mathbf{w}-\mathbf{u}_1)\cdot\mathbf{n}\,\mathrm{d}\mathbf{n}=\pi|\mathbf{q}|\mathbf{q}\cdot(\mathbf{w}-\mathbf{u}_1)$$

one gets

$$\frac{\mathrm{d}\,T(t)}{\mathrm{d}t} \leqslant -\frac{2m}{3}\mu(1-\mu)\int_{\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}|^3 f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w} + \frac{2m}{3}\mu\int_{\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}|^2|\mathbf{w}-\mathbf{u}_1|f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}.$$

Let us first investigate the second integral. One has as above

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}|^2|\mathbf{w}-\mathbf{u}_1|f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\;\mathrm{d}\mathbf{w} = \int_{\mathbb{R}^3}(\mathbf{v}-\mathbf{u}_1)^2 f(\mathbf{v},t)\int_{\mathbb{R}^3}|\mathbf{w}-\mathbf{u}_1|\,M_1(\mathbf{w})\mathrm{d}\mathbf{w} + \int_{\mathbb{R}^3} f(\mathbf{v},t)\mathrm{d}\mathbf{v}\int_{\mathbb{R}^3}|\mathbf{w}-\mathbf{u}_1|^3 M_1(\mathbf{w})\mathrm{d}\mathbf{w},$$

where we used the fact that $\int_{\mathbb{R}^3}(\mathbf{w}-\mathbf{u}_1)|\mathbf{w}-\mathbf{u}_1|\,M_1(\mathbf{w})\mathrm{d}\mathbf{w}=0$. Thus,

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}|^2|\mathbf{w}-\mathbf{u}_1|f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\;\mathrm{d}\mathbf{w} \leqslant C_1 F(t) \tag{5.6}$$

with

$$C_1 = \max\left\{\int_{\mathbb{R}^3}|\mathbf{w}-\mathbf{u}_1|M_1(\mathbf{w})\mathrm{d}\mathbf{w}\,,\;\frac{\int_{\mathbb{R}^3}|\mathbf{w}-\mathbf{u}_1|^3 M_1(\mathbf{w})\mathrm{d}\mathbf{w}}{\int_{\mathbb{R}^3}|\mathbf{w}-u|^2 M_1(\mathbf{w})\mathrm{d}\mathbf{w}}\right\}$$

is a positive (explicit) constant depending only $M_1$. Moreover, Jensen's inequality gives

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} |\mathbf{q}|^3 f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w} \geqslant \left(\int_{\mathbb{R}^3\times\mathbb{S}^2}|\mathbf{q}|^2 f(\mathbf{v},t)M_1(\mathbf{w})\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\right)^{3/2}. \tag{5.7}$$

Now, combining (5.6) and (5.7) and (5.5) one gets

$$\frac{\mathrm{d}\,F(t)}{\mathrm{d}t} \leqslant -2\mu(1-\mu)F(t)^{3/2} + 2\mu C_1 F(t).$$

It is well–known then that

$$F(t) \leqslant \max\left\{\frac{C_1^2}{(1-\mu)^2}, F(0)\right\} \quad t \geqslant 0$$

and, turning back to $T(t)$ one obtains that

$$\sup_{t\geqslant 0} \int_{\mathbb{R}^3} (\mathbf{v}-\mathbf{u}_1)^2 f(\mathbf{v},t)\mathrm{d}\mathbf{v} < \infty \tag{5.8}$$

provided $\int_{\mathbb{R}^3} \mathbf{v}^2 f_0(\mathbf{v})\mathrm{d}\mathbf{v} < \infty$. Note moreover that the above bound for $T(t)$ is explicitly computable in terms of $f_0$, $C_1$, $\alpha$ and $\beta$.

Now, mass conservation and the $H$-Theorem, together with estimate (5.8), show that, if

$$\int_{\mathbb{R}^3} \left(1+\mathbf{v}^2+|\log f_0(\mathbf{v})|\right) f_0(\mathbf{v})\mathrm{d}\mathbf{v} < \infty \tag{5.9}$$

then

$$\sup_{t\geqslant 0} \int_{\mathbb{R}^3} \left(1+(\mathbf{v}-\mathbf{u}_1)^2+|\log f(\mathbf{v},t)|\right) f(\mathbf{v},t)\mathrm{d}\mathbf{v} < \infty$$

and this implies the weak-compactness in $L^1(\mathbb{R}^3)$ of the family $\{f(\mathbf{v},t)\}_{t\geqslant 0}$. Now, following ref. 22, one gets the weak-convergence towards the equilibrium of $f(\mathbf{v},t)$. Then, using translation continuity one can prove the following strong convergence result (see ref. 21 for details).

**Theorem 5.3.** Let $f_0 \in L^1(\mathbb{R}^3)$ be a distribution function with unit mass satisfying (5.9) and let $f(\mathbf{v},t)$ be the solution to the Cauchy problem (5.1). Then

$$\lim_{t\to\infty} \|f(t)-M\|_{L^1(\mathbb{R}^3)} = 0.$$

**Remark 5.4.** We may conjecture that, as it occurs for the pseudo-Maxwellian approximation,[25] the decay of $\|f(t)-M\|_{L^1(\mathbb{R}^3)}$ towards 0 is exponential (with an explicit rate).

## 6. CONCLUSIONS

We have proved existence and uniqueness of a collision equilibrium for the dissipative linear Boltzmann equation with general collision kernel. This equilibrium state is a *universal Maxwellian* with the same mass velocity as the field particles background and with a (non-zero) temperature always lower than the one of the background (depending on mass ratio and inelasticity). This results, as early noticed in ref. 25, from the combined effects of momentum and energy exchange with fields particles on the one side and, on the other side, of energy dissipation in the collisions.

We point out that the existence of a Maxwellian equilibrium at non-zero temperature is of primary importance for who wants to derive the hydrodynamic equations for the considered granular flow. This can be done paraphrasing the conclusions of ref. 25 thanks to a suitable Chapman–Enskog procedure.

Moreover, in space homogeneous conditions, the solution to the linear dissipative Boltzmann equation converges towards the equilibrium state as time goes to infinity for any initial datum with finite entropy and temperature. Unfortunately, our convergence result is based upon compactness arguments and therefore, we have not been able to determine the decay rate towards the equilibrium. We may hope that, as it occurs for the pseudo-Maxwellian approximation,[25] the relaxation to equilibrium is exponential. Moreover, the results of the above Appendix show that the Fokker–Planck equation (6.9) is a good approximation of (1.1) when collisions become grazing. Now, it is well–known (see ref. 1) that the solution to (6.9) relaxes to $M$ exponentially with an explicit rate related to $T^{\#}$. This supports us in the belief that the same occurs for the dissipative Boltzmann Eq. (1.1). We may infer that dissipation–dissipation entropy methods should lead to such a result. Work is in progress in this direction.

## APPENDIX

Let us make rigorous the derivation of the Fokker–Planck equation we formally obtained in Section 3. This can be done *a posteriori* using the results of Section 5. Our notations are those of Section 3. The only difference is that, hereafter $\mathcal{Q}_\delta$ denotes the collision operator with kernel:

$$B_\delta(\mathbf{q},\mathbf{n}) = b_\delta(\theta)|\mathbf{q}\cdot\mathbf{n}|$$

whereas in Section 3 we consider a re-scaled kernel (see Remark 3.1). Actually, in Section 3 we were concerned with an approximation of the

*Boltzmann collision operator* whereas in this appendix, we approximate the *solution to the Boltzmann equation* as collisions become grazing. This will appear clearly in the sequel. Let $f_0$ be a non-negative distribution function. For the sake of simplicity, we assume in this section that $f_0$ is regular (in a sense we will make more precise further) and consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial f_\delta}{\partial t}(\mathbf{v},t) = \mathcal{Q}_\delta(f_\delta)(\mathbf{v},t) \quad t>0, \quad \mathbf{v}\in\mathbb{R}^3 \\ f_\delta(\mathbf{v},0) = f_0(\mathbf{v}) \end{cases} \tag{6.1}$$

As we told it in the Section 5, such a problem (6.1) admits a (unique) weak-solution which satisfies

$$\begin{aligned} -\int_0^\infty \mathrm{d}t \int_{\mathbb{R}^3} f_\delta(\mathbf{v},t)\partial_t\psi(\mathbf{v},t)\mathrm{d}\mathbf{v} - \int_{\mathbb{R}^3} f_0(\mathbf{v})\psi(\mathbf{v},0)\mathrm{d}\mathbf{v} \\ = \frac{1}{2\pi}\int_0^\infty \mathrm{d}t \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n}) f_\delta(\mathbf{v},t) M_1(\mathbf{w}) \\ \cdot\left[\psi(\mathbf{v}^\star,t) - \psi(\mathbf{v},t)\right]\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}. \end{aligned} \tag{6.2}$$

for any $\psi\in C^1_{2,c}([0,+\infty[\times\mathbb{R}^3)$, i.e., $\psi$ is continuously differentiable with compact support in $[0,+\infty[$ and twice continuously differentiable in $\mathbb{R}^3$. Recall that

$$I_\delta = \int_0^{\pi/2} b_\delta(\theta)\cos^3\theta\,\sin\theta\,\mathrm{d}\theta \sim \frac{\delta^4}{2\pi} \quad \text{as} \quad \delta\sim 0. \tag{6.3}$$

We introduce the following time-scaling

$$g_\delta(\mathbf{v},t) = f_\delta(\mathbf{v},\delta^{-4}t).$$

Then, considering a test-function of the form $\psi_\delta(\mathbf{v},t) = \psi(\mathbf{v},\delta^{-4}t)$ into (6.2) leads to

$$-\int_0^\infty \mathrm{d}t \int_{\mathbb{R}^3} g_\delta(\mathbf{v},t)\partial_t\psi(\mathbf{v},t)\mathrm{d}\mathbf{v} - \int_{\mathbb{R}^3} f_0(\mathbf{v})\psi(\mathbf{v},0)\mathrm{d}\mathbf{v}$$

$$= \frac{1}{2\pi\delta^4}\int_0^\infty \mathrm{d}t \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n}) g_\delta(\mathbf{v},t) M_1(\mathbf{w}) \cdot \left[\psi(\mathbf{v}^\star,t)-\psi(\mathbf{v},t)\right]\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}. \tag{6.4}$$

The key point of the approximation procedure is the following

**Proposition 6.1.** There exists a non-negative function $g:[0,+\infty[\to L^1(\mathbb{R}^3)$ and a subsequence, still denoted $(g_\delta)_{\delta\geqslant 0}$ such that $g_\delta$ converges weakly in $L^1_{loc}([0,+\infty[,L^1(\mathbb{R}^3))$ towards $g$ as $\delta$ goes to zero.

We leave the proof to Proposition 6.1 to the end of this appendix and explain now how to derive the Fokker-Plank equation from it. Inserting the expansion (3.2) into (6.4) leads to the following approximation:

$$
\begin{aligned}
&-\int_0^\infty \mathrm{d}t\int_{\mathbb{R}^3} g_\delta(\mathbf{v},t)\partial_t\psi(\mathbf{v},t)\mathrm{d}\mathbf{v} - \int_{\mathbb{R}^3} f_0(\mathbf{v})\psi(\mathbf{v},0)\mathrm{d}\mathbf{v}\\
&=\int_0^\infty \mathbf{J}^1_\delta(t)\mathrm{d}t + \int_0^\infty \mathbf{J}^2_\delta(t)\mathrm{d}t + \mathbf{R}_\delta\\
&=-\frac{2\alpha(1-\beta)}{2\pi\delta^4}\int_0^\infty \mathrm{d}t\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n})(\mathbf{q}\cdot\mathbf{n}) g_\delta(\mathbf{v},t)M_1(\mathbf{w})\nabla_{\mathbf{v}}\psi(\mathbf{v},t)\\
&\quad\cdot\mathbf{n}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}\\
&\quad+\frac{(2\alpha(1-\beta))^2}{4\pi\delta^4}\int_0^\infty \mathrm{d}t\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(\mathbf{q},\mathbf{n})\,(\mathbf{q}\cdot\mathbf{n})^2\, g_\delta(\mathbf{v},t)\\
&\quad M_1(\mathbf{w})\mathbb{D}^2\psi(\mathbf{v},t)\cdot\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}+\mathbf{R}_\delta,
\end{aligned}
$$

where $\mathbf{R}_\delta=\mathbf{R}_\delta(\psi)$ is a suitable remainder term obtained from the Taylor expansion of $\psi$ (3.2) (see ref. 18 for details). One can prove as in the elastic case (18) that

$$\lim_{\delta\to 0}\mathbf{R}_\delta(\psi)=0 \tag{6.5}$$

for any test-function $\psi\in C^1_{2,c}([0,+\infty[\times\mathbb{R}^3)$. As in Section 3 one can show that

$$\int_0^\infty \mathbf{J}^1_\delta(t)\mathrm{d}t = -I_\delta\frac{2\alpha(1-\beta)}{\delta^4}\int_0^\infty \mathrm{d}t\int_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{q}|^2 g_\delta(\mathbf{v},t)M_1(\mathbf{w})\nabla_{\mathbf{v}}\psi(\mathbf{v},t)\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}$$

Consequently, thanks to Proposition 6.1

$$\lim_{\delta\to 0} \mathbf{J}^1_\delta(t)\,\mathrm{d}t = -\frac{\alpha(1-\beta)}{\pi}\int_0^\infty \mathrm{d}t \int_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{q}|^2 g(\mathbf{v},t)M_1(\mathbf{w})\nabla_\mathbf{v}\psi(\mathbf{v},t)\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}. \tag{6.6}$$

We proceed in the same way for $\mathbf{J}^2_\delta$ and one sees that

$$\lim_{\delta\to 0}\int_0^\infty \mathbf{J}^2_\delta(t)\mathrm{d}t = \frac{\alpha^2(1-\beta)^2}{2\pi}\int_0^\infty \mathrm{d}t \int_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{q}|^3 g(\mathbf{v},t)M_1(\mathbf{w})\mathbb{D}^2\psi(\mathbf{v},t) \cdot \mathbf{Diag}[0,1,1]\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}. \tag{6.7}$$

Then, combining (6.4), (6.6) and (6.7), one sees that the weak limit $g$ satisfies

$$-\int_0^\infty \mathrm{d}t \int_{\mathbb{R}^3} g(\mathbf{v},t)\partial_t\psi(\mathbf{v},t)\mathrm{d}\mathbf{v} - \int_{\mathbb{R}^3} f_0(\mathbf{v})\psi(\mathbf{v},0)\mathrm{d}\mathbf{v}$$

$$= -\frac{1}{2\pi}\int_0^\infty \mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}\mathbf{v}\nabla_\mathbf{v}\psi(\mathbf{v},t)\cdot\int_{\mathbb{R}^3}|\mathbf{v}-\mathbf{w}|^3\mathbb{S}(\mathbf{v}-\mathbf{w})\{\kappa M_1(\mathbf{w})\nabla_\mathbf{v} g(\mathbf{v},t) + (\kappa+\mu)g(\mathbf{v},t)\nabla_\mathbf{w} M_1(\mathbf{w})\}\mathrm{d}\mathbf{w}, \tag{6.8}$$

where $\kappa$ and $\mu$ are defined in Section 3. It is not difficult now to recognize in 6.8 the weak formulation of the following Cauchy problem

$$\begin{cases} \dfrac{\partial g}{\partial t}(\mathbf{v},t) = \dfrac{1}{2\pi}\mathcal{Q}_{\mathrm{FP}}(g)(\mathbf{v},t) & t\geqslant 0,\quad \mathbf{v}\in\mathbb{R}^3 \\ g(\mathbf{v},0) = f_0(\mathbf{v}) \end{cases} \tag{6.9}$$

where the Fokker–Planck collision operator is given by (3.9). It is well-known that problem (6.9) admits a (unique) non-negative weak solution $g$. We proved the following approximation result.

**Theorem 6.2.** There exists a subsequence, still denoted $g_\delta$, such that

$$g_\delta \underset{\delta\to 0}{\rightharpoonup} g \text{ weakly in } L^1_{loc}([0,+\infty[,\ L^1(\mathbb{R}^3))$$

where $g$ is a weak solution to the Fokker–Planck equation (3.9) with regular initial data $f_0$.

It remains now to prove Proposition 6.1. Clearly, it is enough to prove the following uniform estimate

$$\sup_{\delta\geqslant 0, t\geqslant 0} \int_{\mathbb{R}^3} (1+(\mathbf{v}-\mathbf{u}_1)^2+|\log g_\delta(\mathbf{v},t)|)g_\delta(\mathbf{v},t)\mathrm{d}\mathbf{v}<\infty. \tag{6.10}$$

Now the estimate

$$\sup_{\delta\geqslant 0, t\geqslant 0} \int_{\mathbb{R}^3} (1+|\log g_\delta(\mathbf{v},t)|)g_\delta(\mathbf{v},t)\mathrm{d}\mathbf{v}<\infty,$$

follows form the $H$-Theorem applied to $f_\delta(t)$. Note that the $H$-Theorem turns to be valid for the collision kernel $B_\delta(\cdot)$ since the equilibrium state is universal (see Remark 4.2). Now to prove the remaining estimate, we proceed as we did in Section 5 to derive formula (5.8). We only sketch here the main changes. Let

$$T_\delta(t)=\frac{m}{3}\int_{\mathbb{R}^3} g_\delta(\mathbf{v},t)(\mathbf{v}-\mathbf{u}_1)^2\mathrm{d}\mathbf{v}=\frac{m}{3}\int_{\mathbb{R}^3} f_\delta(\mathbf{v},\delta^{-4}t)(\mathbf{v}-\mathbf{u}_1)^2\mathrm{d}\mathbf{v}.$$

Then,

$$\frac{\mathrm{d}\,T_\delta(t)}{\mathrm{d}t}=\frac{m\delta^4}{6\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\delta(\mathbf{q}\cdot\mathbf{n})f_\delta(\mathbf{v},\delta^{-4}t)M_1(\mathbf{w}) \left\{(\mathbf{v}^\star-\mathbf{u}_1)^2-(\mathbf{v}-\mathbf{u}_1)^2\right\}\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n}.$$

Now, the only supplementary difficulty lies in the computation of

$$\int_{\mathbb{S}^2} |\mathbf{q}\cdot\mathbf{n}|^2 B_\delta(v\mathbf{q}\cdot\mathbf{n})\mathrm{d}\mathbf{n}=2\pi I_\delta|\mathbf{q}|^3 \quad\text{and}$$

$$\int_{\mathbb{S}^2} B_\delta(\mathbf{q}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n})(\mathbf{w}-\mathbf{u}_1)\cdot\mathbf{n}\,\mathrm{d}\mathbf{n}\leqslant 2\pi I_\delta|\mathbf{q}|^2|\mathbf{w}-\mathbf{u}_1|$$

and the conclusion follows as in Section 5 since $\lim_{\delta\to 0}2\pi\delta^4 I_\delta=1$.

**Remark 6.3.** The derivation of the Fokker–Planck equation (6.9) calls for comments. As already mentioned, it has been made possible because of the nature of inelastic collision mechanism. Indeed, since the loss of the relative velocity due to inelasticity only occurs for the normal component while the tangential component of the relative velocity remains unchanged. Then, in the grazing collision limit the *local energy is conserved* since the difference between pre- and post-collisional energies are proportional to a function of the scattering angle, which vanishes as the collision is grazing (independently of $\beta$).

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